ESTIMATING NORMS OF COMMUTATORS

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Abstract. We discuss a general method of finding bounds on the norm of a commutator of an operator and a function of a normal operator. As an application we find new bounds on the norm of a commutator with a square root.

1. Norms of Commutators and functional calculus

For f a continuous function $\mathbb{R}$ that is periodic, period 2π always assumed, then we will have need to apply it via functional calculus to both hermitian and unitary elements, in the latter case by interpreting f as a function on the circle. Just to be clear, we introduce the notation

$$f[V] = \tilde{f}(V)$$

for V any unitary element in a unital C^*-algebra $\mathcal{A}$, where

$$\tilde{f}(z) = f\left(-i\log(z)\right)$$

for any z of modulus one. For example

$$\cos[V] = \tfrac{1}{2}V^* + \tfrac{1}{2}V.$$

It is trivial to prove that when an element A in $\mathcal{A}$ commutes with V then A commutes with $f[V]$. We will need good estimates that quantify the statement that when A almost commutes with V then it also almost commutes with $f[V]$.

The only norm on $[A, V]$ we really care about is the operator norm, i.e. the norm on $\mathcal{A}$, that we denote $\|\cdot\|$. As to functions f that are periodic, we need

$$\|f\|_\infty = \sup_{-\pi\leq x\leq\pi} |f(x)|$$

and, whenever f has Fourier series converging absolutely, we use

$$\|f\|_F = \|\hat{f}\|_1$$

the ℓ^1 norm of the Fourier series. We use $\mathcal{U}(A)$ for the group of unitaries in $\mathcal{A}$.

Definition 1.1. Suppose f is continuous and periodic. Define $\eta_f : [0,\infty) \to [0,\infty)$ by

$$f_\eta(\delta) = \sup\left\{ \|[f[V], A]\| \,\middle|\, V \in \mathcal{U}(A), \|A\| \leq 1, \|[V, A]\| \leq \delta \right\}$$

and the supremum is taken over every possible C^*-algebra $\mathcal{A}$ and taking V and A in $\mathcal{A}$.

There is a general trend where results about commutators are related to continuity results involving the functional calculus. See [1], for example. In the case of unitaries there is an easy connection between the two topics.

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Lemma 1.2. *For f that is continuous and periodic, if V and V_1 are unitaries then*

$$\|f[V]-f[V_1]\| \leq \eta_f\left(\|V-V_1\|\right).$$

Proof. Notice

$$\left\|\left[\left(\begin{array}{cc} 0 & 1\\ 1 & 0 \end{array}\right),\left(\begin{array}{cc} 0 & V\\ V_1 & 0 \end{array}\right)\right]\right\| = \|V-V_1\|$$

and

$$\left\|\left[\left(\begin{array}{cc} 0 & 1\\ 1 & 0 \end{array}\right),\left(\begin{array}{cc} 0 & f[V]\\ f[V_1] & 0 \end{array}\right)\right]\right\| = \|f[V]-f[V_1]\|$$

so this is an easy calculation. □

The following generalizes a trick in Pedersen's work on commutators and square roots, [4, Lemma 6.2].

Lemma 1.3. *Suppose f, g and h are continuous and periodic, that g' has absolutely convergent Fourier series. If $f=g+h$ then*

$$\eta_f(\delta) \leq m\delta + b$$

where

$$m = \|g'\|_F$$

and

$$b = 2\min_{\lambda\in\mathbb{C}} \|h-\lambda\|_\infty .$$

When h is real valued then $b=\max(h)-\min(h)$.

Remark. In the special case where $h=0$ we recover the folk theorem that says

$$\|[g[V],A]\| \leq \|g'\|_F \, \|[V,A]\| \tag{1.1}$$

for any unitary V and any operator A, now without norm restriction because the two sides are homogeneous in A.

Proof. Suppose $\|A\|\leq 1$ and V is unitary. Since

$$\|[f[V],A]\| \leq \|[g[V],A]\| + \|[h[V],A]\|$$

and

$$\|[h[V],A]\| = \|[h[V]+\lambda I,A]\| = \|[(h+\lambda)[V],A]\|$$

it suffices to prove equation (1.1) and

$$\|[h[V],A]\| \leq 2\|h\|_\infty . \tag{1.2}$$

We know

$$g(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

where $\sum |na_n| < \infty$ and so

$$\begin{aligned}\|[g[V],A]\| &= \left\| \left[\sum_{n=-\infty}^{\infty} a_n V^n, A \right] \right\| \\ &\leq \sum_{n=-\infty}^{\infty} |a_n| \, \|[V^n, A]\| \\ &\leq \sum_{n=-\infty}^{\infty} |na_n| \, \|[V, A]\| \\ &= \|g'\|_F \, \|[V,A]\| .\end{aligned}$$

The spectral theorem tells us $\|h[V]\| \leq \|h\|_\infty$ and so

$$\|[h[V],A]\| = \|h[V]A - h[V]A\| \leq 2\,\|h[V]\|\,\|A\| \leq 2\,\|h\|_\infty .$$

□

As an example, we attack the square root function $f(x) = \sqrt{x}$. However, this is for $0 \leq H \leq 1$ replacing V so is about γ_f not η_f, where γ_f we now define for working with functional calculus of positive contractions.

Definition 1.4. Suppose f is continuous on $[0,1]$. Define $\eta_f : [0,\infty) \to [0,\infty)$ by

$$f_\eta(\delta) = \sup \left\{ \|[f(H),A]\| \,\middle|\, 0 \leq H \leq 1, \|A\| \leq 1, \|H\| \leq 1, \|[H,A]\| \leq \delta \right\}$$

and the supremum is taken over every possible C^*-algebra $\mathcal{A}$ and taking H and A in $\mathcal{A}$.

Lemma 1.5. *Suppose f, g and h are continuous on $[0,1]$ and that g is analytic, with power series*

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

If $f = g + h$ then

$$\eta_f(\delta) \leq m\delta + b$$

where

$$m = \sum_{n=0}^{\infty} |na_n|$$

and

$$b = 2 \min_{\lambda \in \mathbb{C}} \|h - \lambda\|_\infty .$$

Proof. We know $\sum |na_n| < \infty$ and so

$$\begin{aligned}\|[g(H),A]\| &= \left\| \left[\sum_{n=0}^{\infty} a_n H^n , A \right] \right\| \\ &= \left\| \sum_{n=0}^{\infty} a_n \left[H^n , A \right] \right\| \\ &\leq \sum_{n=0}^{\infty} |a_n| \, \|[H^n, A]\| \\ &\leq \sum_{n=0}^{\infty} |na_n| \, \|[H,A]\| \, .\end{aligned}$$

It is not clear who first asserted the following, but it appears in [5]. □

Conjecture 1.6. *For* $f(x) = \sqrt{x}$ *we have* $\gamma_f(\delta) = \sqrt{\delta}$*. Equivalently*

$$\left\| \left[H^{\frac{1}{2}}, A \right] \right\| \leq \|[H,A]\|^{\frac{1}{2}}$$

wherever $0 \leq H \leq 1$ *and* $\|A\| \leq 1$*.*

For any a greater than 0 and at most 1 let g be the Taylor expansion of f at a,

$$g(x) = \frac{1}{2\sqrt{a}}(x-a) + \sqrt{a}.$$

and $h = f - g$,

$$h(x) = \sqrt{x} - \frac{1}{2\sqrt{a}}(x-a) - \sqrt{a}.$$

Clearly $\max(h) = h(a) = 0$ and the minimum occurs at either $x = 0$ or $x = 1$, where the values are

$$h(0) = -\tfrac{1}{2}\sqrt{a}$$

and

$$h(1) = 1 - \frac{1}{2\sqrt{a}} - \frac{\sqrt{a}}{2}.$$

For $\frac{1}{4} \leq a \leq 1$ we find

$$\min(h) = -\tfrac{1}{2}\sqrt{a}.$$

Therefore,

$$\eta_f(\delta) \leq \frac{1}{2\sqrt{a}}\delta + \tfrac{1}{2}\sqrt{a} \tag{1.3}$$

for $\frac{1}{4} \leq a \leq 1$, which is very interesting since at $\delta = a$ the right hand side is $\sqrt{a}$. We have proven a special case of the conjecture, which we state as a lemma.

Lemma 1.7. *When* $0 \leq H \leq 1$ *and* $\|A\| \leq 1$ *and* $\|[H,A]\| \geq \frac{1}{4}$*, we have*

$$\left\| \left[H^{\frac{1}{2}}, A \right] \right\| \leq \|[H,A]\|^{\frac{1}{2}} .$$

Pedersen uses the following easy lemma.

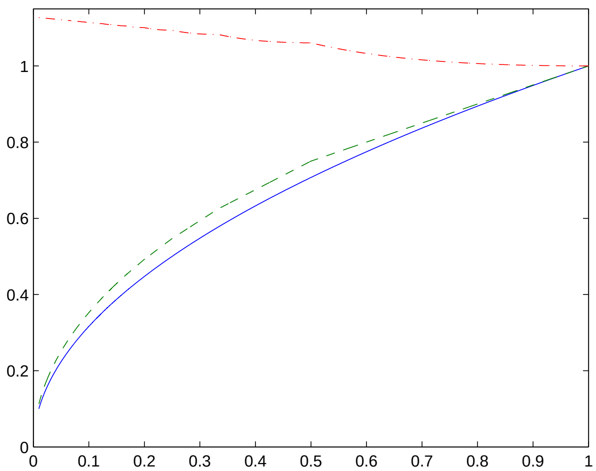

FIGURE 1.1. Bound on $\left\|\left[H^{\frac{1}{2}},A\right]\right\|$ for varying values of $\|[H,A]\|$ as found by Pedersen, shown as a dashed line. The solid curve is $\sqrt{\delta}$. The top curve is the ratio of the bound to $\sqrt{\delta}$.

Lemma 1.8. *If f_1 is continuous on $[0,1]$ and we set*

$$f_2(x) = 1 - f_1(1-x)$$

that $\gamma_{f_1} = \gamma_{f_2}$.

His proof of the inequality

$$\left\|\left[H^{\frac{1}{2}},A\right]\right\| \leq \frac{2}{\sqrt{\pi}}\,\|[H,A]\|^{\frac{1}{2}} \tag{1.4}$$

(notice $2\pi^{\frac{1}{2}} \approx 1.128$) in [4, Lemma 6.2] invokes Lemma 1.5 infinitely many times, as g ranges over the Taylor polynomials for $f(x) = 1 - \sqrt{1-x}$ expanded at 0. While (1.4) is the statement

$$\gamma_f(\delta) \leq \frac{2}{\sqrt{\pi}}\delta^{\frac{1}{2}}$$

what he actually proves is a bound that is significantly smaller for δ close to 1. Indeed, he showed γ_f to be bounded by the function shown in Figure 1.1

The minimum of all these lines does not lead to an easy formula, so we state our best theorem regarding the square root in terms on a ploted function.

Theorem 1.9. *If $0 \leq H \leq 1$ and $\|A\| \leq 1$ then*

$$\left\|\left[H^{\frac{1}{2}},A\right]\right\| \leq \gamma_0\left(\|[H,A]\|\right)$$

where γ is the function illustrated in Figure 1.2.

Proof. We simply combine all the linear bounds in [4, Lemma 6.2] with Lemma 1.7. □

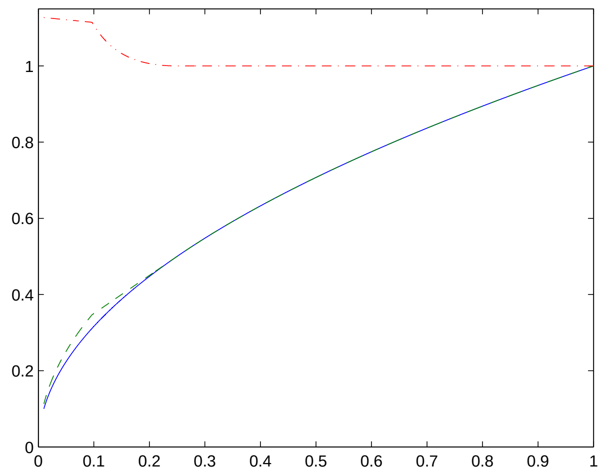

FIGURE 1.2. Bound on $\left\|\left[H^{\frac{1}{2}}, A\right]\right\|$ for varying values of $\delta = \|[H, A]\|$ as improved by the inequalities (1.3). The solid curve is $\sqrt{\delta}$. The dashed curve is the upper bound $\gamma_0(\delta)$ of Theorem 1.9. The top curve is $\gamma_0(\delta)/\sqrt{\delta}$.

2. EXAMPLES INVOLVING FUNCTIONS ON THE CIRCLE

There is a desire, driven by investigations in physics [3], to get quantitative results regarding almost commuting matrices. The Bott index for almost commuting matrices depends on the functional calculus of unitary matrices. Quantitative studies of the Bott index require triples of functions

$$f, g, h : \mathbb{T}^2 \to \mathbb{R}^3$$

with certain topological properties. Having a method for dealing with $\|[f[V], U]\|$for a pair of unitary elements was the primary motivation for the present paper.

Corollary 2.1. *If f has uniformly convergent Fourier series,*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

then

$$\eta_f(\delta) \leq 2 \sum_{n=-\infty}^{\infty} |a_n|.$$

and

$$\eta_f(\delta) \leq \delta \sum_{n=-N}^{N} |n a_n| + 2 \sum_{n=N+1}^{\infty} \left(|a_n| + |a_{-n}|\right). \tag{2.1}$$

Proof. If we take

$$g(x) = \sum_{n=-N}^{N} a_n e^{inx}$$

and apply Lemma 1.3 we obtain (2.1). For the other we set g to 0. ∎

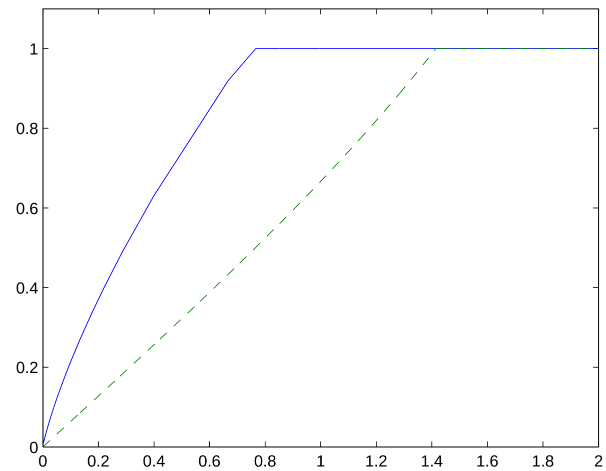

FIGURE 2.1. Bounds on $\|[f[V], A]\|$ for varying values of $\delta = \|[V, A]\|$ for f a triangle wave. The solid line is an upper bound and the dashed line is a lower bound.

Example 2.2. Consider the triangle wave

$$f(x) = \begin{cases} 1 + \frac{2}{\pi}x & -\pi \leq x \leq 0 \\ 1 - \frac{2}{\pi}x & 0 \leq x \leq \pi \end{cases}$$

we have $a_{2n} = 0$ and

$$a_{2n-1} = \frac{8}{\pi^2}\frac{1}{(2n-1)^2}.$$

Using Corollary 2.1 we get the bound on η_f as indicated in Figure 2.1. Slightly better estimates are possible if we exactly compute the min and max of the difference between f and its trigonometric polynomial approximations. We could also eliminate the corners by interpolating with trig polynomials between the truncated Fourier series.

The triangle wave is, up to scaling, the function used in [2] as one of the functions defining the Bott invariant. To see how well we are doing in bounding $\|[f[V], A]\|$ we consider a crude lower bound.

Lemma 2.3. *If f is periodic and continuous then for any $\delta < 2$ we have*

$$\eta_f(\delta) \geq \max\left\{ |f(x_2) - f(x_1)| \;\Big|\; |x_2 - x_1| \leq 2\arcsin\left(\tfrac{\delta}{2}\right) \right\}.$$

The follows easily from examining the commutator of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} e^{ix_2} & 0 \\ 0 & e^{ix_2} \end{pmatrix}$$

and

$$f\left[\begin{pmatrix} e^{ix_2} & 0 \\ 0 & e^{ix_2} \end{pmatrix}\right] = \begin{pmatrix} f(x_1) & 0 \\ 0 & f(x_2) \end{pmatrix}.$$

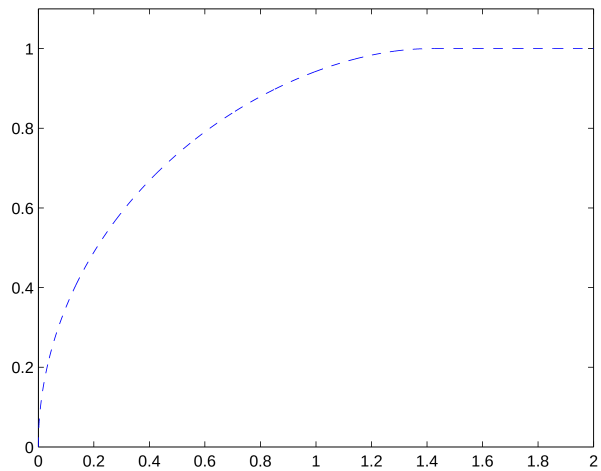

FIGURE 2.2. A lower bound on $\|[h[V], A]\|$ where h is the bump function on the circle defined in Equation 2.2.

Thus in the example of the triangle wave, we could may have considerable room to improve our estimate. However, for the purposes of "quantitative K-theory" involving the Bott index, this f shows limited potential, as its companion functions g and h are not so nice. That is, in the Bott index definition as in [2] we also need

$$h(x) = \begin{cases} \sqrt{1 - \frac{4}{\pi^2}x^2} & \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ 0 & \text{if } x \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \end{cases} \tag{2.2}$$

and η_h tends to zero rather slowly. The crude lower bound from Lemma 2.3 is shown in Figure 2.2. This is one reason for the switch to a different triple of functions f, g and h in [3]. Commutators involving the functions in the new and improved Bott index will be analyzed elsewhere.

3. Acknowledgements

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